The zeta function of H^4 counting ideals

1 Presentation

H^4 has presentation

$$\left\langle x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4 \;\middle|\; \begin{array}{c} [x_1, y_1] = z_1, [x_2, y_2] = z_2, \\ [x_3, y_3] = z_3, [x_4, y_4] = z_4 \end{array} \right\rangle.$$

H^4 has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{H^4,p}(s) &= \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(s-5)\zeta_p(s-6) \\ &\quad \times \zeta_p(s-7)\zeta_p(3s-8)^4\zeta_p(5s-9)\zeta_p(7s-10)\zeta_p(8s-18) \\ &\quad \times \zeta_p(9s-11)\zeta_p(10s-20)\zeta_p(11s-27)W(p,p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned}
&1 - 6X^{8}Y^{5} + 5X^{9}Y^{5} + 4X^{8}Y^{7} - 8X^{9}Y^{7} + 3X^{10}Y^{7} + 4X^{16}Y^{8} - 8X^{17}Y^{8} \\
&+ 3X^{18}Y^{8} - X^{8}Y^{9} + 3X^{9}Y^{9} - 3X^{10}Y^{9} - X^{16}Y^{10} + 5X^{17}Y^{10} - 6X^{18}Y^{10} \\
&+ X^{19}Y^{10} - X^{24}Y^{11} + 3X^{25}Y^{11} - 3X^{26}Y^{11} + 8X^{17}Y^{12} - 10X^{18}Y^{12} \\
&+ 3X^{19}Y^{12} + 8X^{25}Y^{13} - 10X^{26}Y^{13} + 3X^{27}Y^{13} - 5X^{17}Y^{14} + 15X^{18}Y^{14} \\
&- 9X^{19}Y^{14} - 19X^{25}Y^{15} + 43X^{26}Y^{15} - 24X^{27}Y^{15} + 2X^{28}Y^{15} - 3X^{18}Y^{16} \\
&+ 5X^{19}Y^{16} - X^{20}Y^{16} - 5X^{33}Y^{16} + 15X^{34}Y^{16} - 9X^{35}Y^{16} + 8X^{25}Y^{17} \\
&- 30X^{26}Y^{17} + 32X^{27}Y^{17} - 7X^{28}Y^{17} - X^{29}Y^{17} + 8X^{33}Y^{18} - 30X^{34}Y^{18} \\
&+ 32X^{35}Y^{18} - 7X^{36}Y^{18} - X^{37}Y^{18} + 3X^{26}Y^{19} - 9X^{27}Y^{19} + 7X^{28}Y^{19} \\
&- 3X^{42}Y^{19} + 5X^{43}Y^{19} - X^{44}Y^{19} - 3X^{33}Y^{20} + 8X^{34}Y^{20} - 17X^{35}Y^{20} \\
&+ 15X^{36}Y^{20} - 3X^{37}Y^{20} - 3X^{27}Y^{21} + X^{28}Y^{21} + X^{29}Y^{21} + 3X^{42}Y^{21} \\
&- 9X^{43}Y^{21} + 7X^{44}Y^{21} + 4X^{34}Y^{22} - 12X^{35}Y^{22} - 2X^{36}Y^{22} + 13X^{37}Y^{22} \\
&- 5X^{38}Y^{22} + 4X^{42}Y^{23} - 12X^{43}Y^{23} - 2X^{44}Y^{23} + 13X^{45}Y^{23} - 5X^{46}Y^{23} \\
&+ 9X^{35}Y^{24} - 10X^{36}Y^{24} - 10X^{37}Y^{24} + 9X^{38}Y^{24} - 3X^{51}Y^{24} + X^{52}Y^{24} \\
&+ X^{53}Y^{24} - 3X^{42}Y^{25} + 18X^{43}Y^{25} - 16X^{44}Y^{25} - 16X^{45}Y^{25} + 18X^{46}Y^{25} \\
&- 3X^{47}Y^{25} + X^{36}Y^{26} + X^{37}Y^{26} - 3X^{38}Y^{26} + 9X^{51}Y^{26} - 10X^{52}Y^{26} \\
&- 10X^{53}Y^{26} + 9X^{54}Y^{26} - 5X^{43}Y^{27} + 13X^{44}Y^{27} - 2X^{45}Y^{27} - 12X^{46}Y^{27}
\end{aligned}$$

$$
\begin{aligned}
&+4X^{47}Y^{27}-5X^{51}Y^{28}+13X^{52}Y^{28}-2X^{53}Y^{28}-12X^{54}Y^{28}+4X^{55}Y^{28}\\
&+7X^{45}Y^{29}-9X^{46}Y^{29}+3X^{47}Y^{29}+X^{60}Y^{29}+X^{61}Y^{29}-3X^{62}Y^{29}\\
&-3X^{52}Y^{30}+15X^{53}Y^{30}-17X^{54}Y^{30}+8X^{55}Y^{30}-3X^{56}Y^{30}-X^{45}Y^{31}\\
&+5X^{46}Y^{31}-3X^{47}Y^{31}+7X^{61}Y^{31}-9X^{62}Y^{31}+3X^{63}Y^{31}-X^{52}Y^{32}\\
&-7X^{53}Y^{32}+32X^{54}Y^{32}-30X^{55}Y^{32}+8X^{56}Y^{32}-X^{60}Y^{33}-7X^{61}Y^{33}\\
&+32X^{62}Y^{33}-30X^{63}Y^{33}+8X^{64}Y^{33}-9X^{54}Y^{34}+15X^{55}Y^{34}-5X^{56}Y^{34}\\
&-X^{69}Y^{34}+5X^{70}Y^{34}-3X^{71}Y^{34}+2X^{61}Y^{35}-24X^{62}Y^{35}+43X^{63}Y^{35}\\
&-19X^{64}Y^{35}-9X^{70}Y^{36}+15X^{71}Y^{36}-5X^{72}Y^{36}+3X^{62}Y^{37}-10X^{63}Y^{37}\\
&+8X^{64}Y^{37}+3X^{70}Y^{38}-10X^{71}Y^{38}+8X^{72}Y^{38}-3X^{63}Y^{39}+3X^{64}Y^{39}\\
&-X^{65}Y^{39}+X^{70}Y^{40}-6X^{71}Y^{40}+5X^{72}Y^{40}-X^{73}Y^{40}-3X^{79}Y^{41}\\
&+3X^{80}Y^{41}-X^{81}Y^{41}+3X^{71}Y^{42}-8X^{72}Y^{42}+4X^{73}Y^{42}+3X^{79}Y^{43}\\
&-8X^{80}Y^{43}+4X^{81}Y^{43}+5X^{80}Y^{45}-6X^{81}Y^{45}+X^{89}Y^{50}.
\end{aligned}
$$

$\zeta^{\triangleleft}_{H^4}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{H^4,p}(s)\Big|_{p\to p^{-1}} = p^{66-20s}\zeta^{\triangleleft}_{H^4,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{H^4}(s)$ is 8, with a simple pole at $s=8$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$
\begin{aligned}
&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(s-5)\zeta_p(s-6)\zeta_p(s-7)\zeta_p(3s-8)^4\\
&\times\zeta_p(5s-9)\zeta_p(7s-10)\zeta_p(8s-18)\zeta_p(9s-11)\zeta_p(10s-20)\zeta_p(11s-27)\\
&\times W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})W_4(p,p^{-s})W_5(p,p^{-s})
\end{aligned}
$$

where

$$
\begin{aligned}
W_1(X,Y) &= 1-3X^{26}Y^{11},\\
W_2(X,Y) &= -3-X^{18}Y^{8},\\
W_3(X,Y) &= -1+X^{9}Y^{5}-3X^{18}Y^{10}-3X^{27}Y^{15},\\
W_4(X,Y) &= -3-X^{10}Y^{7},\\
W_5(X,Y) &= -1+X^{8}Y^{9}.
\end{aligned}
$$

The ghost is unfriendly.

6 Natural boundary

$\zeta^{\triangleleft}_{H^4}(s)$ has a natural boundary at $\Re(s) = 26/11$, and is of type I.